Metadata of the chapter that will be visualized in SpringerLink

Book Title	Lie Theory and Its Applications in Physics	
Series Title		
Chapter Title	Nested Bethe Ansatz for RTT–Algebra of $U_q\big(\mathrm{sp}(2n)\big)$ Type	
Copyright Year	2020	
Copyright HolderName	Springer Nature Singapore Pte Ltd.	
Corresponding Author	Family Name	**Burdík**
	Particle	
	Given Name	**.**
	Prefix	
	Suffix	
	Role	
	Division	Faculty of Nuclear Sciences and Physical Engineering
	Organization	CTU
	Address	Trojanova 13, Prague, Czech Republic
	Email	cestmir.burdik@fjfi.cvut.cz
Author	Family Name	**Navrátil**
	Particle	
	Given Name	**O.**
	Prefix	
	Suffix	
	Role	
	Division	Faculty of Transportation Sciences
	Organization	CTU
	Address	Na Florenci 25, Prague, Czech Republic
	Email	navraond@fd.cvut.cz
Abstract	We study the highest weight representations of the RTT–algebras for the R–matrix of $\mathrm{sp}_q(2n)$ type by the nested algebraic Bethe ansatz. It is a generalization of our study for R–matrix of $\mathrm{sp}(2n)$ and $\mathrm{so}(2n)$ type.	

Nested Bethe Ansatz for RTT–Algebra of $U_q(\mathrm{sp}(2n))$ Type

Č. Burdík and O. Navrátil

Abstract We study the highest weight representations of the RTT–algebras for the R–matrix of $\mathrm{sp}_q(2n)$ type by the nested algebraic Bethe ansatz. It is a generalization of our study for R–matrix of $\mathrm{sp}(2n)$ and $\mathrm{so}(2n)$ type. AQ1

1 Introduction

The formulation of the quantum inverse scattering method, or algebraic Bethe ansatz, by the Leningrad school [1] provides eigenvectors and eigenvalues of the transfer matrix. The latter is the generating function of the conserved quantities of a large family of quantum integrable models. The transfer matrix eigenvectors are constructed from the representation theory of the RTT–algebras. In order to construct these eigenvectors, one should first prepare Bethe vectors, depending on a set of complex variables. The first formulation of the Bethe vectors for the $\mathrm{gl}(n)$–invariant models was given by P.P. Kulish and N.Yu. Reshetikhin in [2] where the nested algebraic Bethe ansatz was introduced. These vectors are given by recursion on the rank of the algebra. Our calculation is some q–generalization of the construction which we published in recent works [3, 4, 6] for the non-deformed case of $\mathrm{sp}(2n)$, $\mathrm{so}(2n)$ and $\mathrm{sp}(4)$.

Our construction of Bethe vectors used the new RTT–algebra $\tilde{\mathcal{A}}_n$ which is defined in Sect. 3 and is not the RTT–subalgebra of $\mathrm{sp}_q(2n)$.

Č. Burdík (✉)
Faculty of Nuclear Sciences and Physical Engineering,
CTU, Trojanova 13, Prague, Czech Republic
e-mail: cestmir.burdik@fjfi.cvut.cz

O. Navrátil
Faculty of Transportation Sciences, CTU, Na Florenci 25, Prague, Czech Republic
e-mail: navraond@fd.cvut.cz

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_22

This algebra has two RTT–subalgebras of $\mathrm{gl}_q(n)$ type and the study of the nested Bethe ansatz for this RTT–algebra is in progress. The simplest case for $n = 2$ was really solved and we will publish in the next paper.

Our construction of Bethe vectors is in any sense a generalization of Reshetikhin's results [7]. Another approach to the nested Bethe ansatz for very special representations of RTT–algebras of $\mathrm{sp}(2n)$ type was given by Martin and Ramas [8].

In this note, due to the lack of space, we omit the proofs of many claims. Mostly, it is possible to prove them similarly as the corresponding claims in [6].

2 Basic Definitions and Notation

Let indices go trough the set $\{\pm 1, \pm 2, \ldots, \pm n\}$. We will denote by $\mathbf{E}_i^k$ the matrices that have all elements equal to zero with the exception of the element on the i-th row and k-th column that is equal to one. Then $\mathbf{I} = \sum\limits_{k=-n}^{n} \mathbf{E}_k^k$ is the unit matrix and $\mathbf{E}_i^k \mathbf{E}_r^s = \delta_r^k \mathbf{E}_i^s$ is valid.

We will consider the R–matrix of $U_q\big(\mathrm{sp}(2n)\big)$ which has the shape

$$\begin{aligned}\mathbf{R}(x) = \frac{1}{\alpha(x)}\Big(&\sum_{i,k;\, i \neq \pm k} \mathbf{E}_i^i \otimes \mathbf{E}_k^k + f(x)\sum_i \mathbf{E}_i^i \otimes \mathbf{E}_i^i \\ &+ f(x^{-1}q^{-n-1})\sum_i \mathbf{E}_i^i \otimes \mathbf{E}_{-i}^{-i} + g(x)\sum_{k<i} \mathbf{E}_k^i \otimes \mathbf{E}_i^k - g(x^{-1})\sum_{i<k} \mathbf{E}_k^i \otimes \mathbf{E}_i^k \\ &- g(xq^{n+1})\sum_{k<i} q^{k-i}\epsilon_i\epsilon_k \mathbf{E}_k^i \otimes \mathbf{E}_{-k}^{-i} + g(x^{-1}q^{-n-1})\sum_{i<k} q^{k-i}\epsilon_i\epsilon_k \mathbf{E}_k^i \otimes \mathbf{E}_{-k}^{-i}\Big)\end{aligned}$$

where $\epsilon_i = \mathrm{sign}(i)$ and

$$f(x) = \frac{xq - x^{-1}q^{-1}}{x - x^{-1}}\,, \qquad g(x) = \frac{x(q - q^{-1})}{x - x^{-1}}\,, \qquad \alpha(x) = 1 + \frac{q - q^{-1}}{x - x^{-1}}\,.$$

This R–matrix satisfies the Yang-Baxter equation

$$\mathbf{R}_{1,2}(x)\mathbf{R}_{1,3}(xy)\mathbf{R}_{2,3}(y) = \mathbf{R}_{2,3}(y)\mathbf{R}_{1,3}(xy)\mathbf{R}_{1,2}(x)$$

and is invertible.

The RTT–algebra of $U_q\big(\mathrm{sp}(2n)\big)$ type is an associative algebra $\mathcal{A}$ with unit, which is generated by $T_k^i(x)$, for which the monodromy operator

$$\mathbf{T}(x) = \sum_{i,k=-n}^{n} \mathbf{E}_i^k \otimes T_k^i(x)$$

fulfills the RTT–equation

$$\mathbf{R}_{1,2}(xy^{-1})\mathbf{T}_1(x)\mathbf{T}_2(y) = \mathbf{T}_2(y)\mathbf{T}_1(x)\mathbf{R}_{1,2}(xy^{-1})\,.$$

From the invertibility of the R–matrix we have that the operator

$$H(x) = \mathrm{Tr}\big(\mathbf{T}(x)\big) = \sum_{i=-n}^{n} T_i^i(x)$$

fulfills the equation $H(x)H(y) = H(y)H(x)$ for any x and y.

We suppose that in the representation space $\mathcal{W}$ of the RTT–algebra $\mathcal{A}$ there exists a vacuum vector $\omega \in \mathcal{W}$, for which $\mathcal{W} = \mathcal{A}\omega$ and

$$T_k^i(x)\omega = 0 \quad \text{pro} \quad i < k\,, \qquad T_i^i(x)\omega = \lambda_i(x)\omega \quad \text{pro} \quad i = \pm 1,\ \pm 2,\ \ldots,\ \pm n\,.$$

In the vector space $\mathcal{W} = \mathcal{A}\omega$, we will look for eigenvectors of $H(x)$.

3 RTT–Algebra $\tilde{\mathcal{A}}_n$

In the RTT–algebra $\mathcal{A}$, we have the RTT–subalgebras $\mathcal{A}^{(+)}$ and $\mathcal{A}^{(-)}$ that are generated by the elements $T_k^i(x)$ and $T_{-k}^{-i}(x)$, where $i,\ k = 1,\ 2,\ \ldots,\ n$. First, we will study the subspace

$$\mathcal{W}_0 = \mathcal{A}^{(+)}\mathbf{A}^{(-)}\omega \subset \mathcal{W} = \mathcal{A}\omega\,.$$

Lemma 1. For any $i,\ k = 1,\ 2,\ \ldots,\ n$ and any $\Omega \in \mathcal{W}_0$ $T_k^{-i}(x)\Omega = 0$ is valid.

Lemma 2. If we denote

$$\mathbf{T}^{(+)}(x) = \sum_{i,k=1}^{n} \mathbf{E}_i^k \otimes T_k^i(x)\,, \qquad \mathbf{T}^{(-)}(x) = \sum_{i,k=1}^{n} \mathbf{E}_{-i}^{-k} \otimes T_{-k}^{-i}(x)\,,$$

then on the space $\mathcal{W}_0$ for any $\epsilon_1,\ \epsilon_2 = \pm$

$$\mathbf{R}_{1,2}^{(\epsilon_1,\epsilon_2)}(xy^{-1})\mathbf{T}_1^{(\epsilon_1)}(x)\mathbf{T}_2^{(\epsilon_2)}(y) = \mathbf{T}_2^{(\epsilon_2)}(y)\mathbf{T}_1^{(\epsilon_1)}(x)\mathbf{R}_{1,2}^{(\epsilon_1,\epsilon_2)}(xy^{-1}) \qquad (1)$$

where

$$\begin{aligned}
\mathbf{R}_{1,2}^{(+,+)}(x) &= \frac{1}{f(x)}\Big(\sum_{i,k=1;\, i\neq k}^{n} \mathbf{E}_i^i \otimes \mathbf{E}_k^k \, + \, f(x)\sum_{i=1}^{n} \mathbf{E}_i^i \otimes \mathbf{E}_i^i \\
&\quad + g(x) \sum_{1\leq k<i\leq n} \mathbf{E}_k^i \otimes \mathbf{E}_i^k - g(x^{-1}) \sum_{1\leq i<k\leq n} \mathbf{E}_k^i \otimes \mathbf{E}_i^k \Big) \\
\mathbf{R}_{1,2}^{(-,-)}(x) &= \frac{1}{f(x)}\Big(\sum_{i,k=1;\, i\neq k}^{n} \mathbf{E}_{-i}^{-i} \otimes \mathbf{E}_{-k}^{-k} + f(x)\sum_{i=1}^{n} \mathbf{E}_{-i}^{-i} \otimes \mathbf{E}_{-i}^{-i} \\
&\quad + g(x) \sum_{1\leq i<k\leq n} \mathbf{E}_{-k}^{-i} \otimes \mathbf{E}_{-i}^{-k} - g(x^{-1}) \sum_{1\leq k<i\leq n} \mathbf{E}_{-k}^{-i} \otimes \mathbf{E}_{-i}^{-k} \Big)
\end{aligned}$$

$$\begin{aligned}
\mathbf{R}_{1,2}^{(+,-)}(x) &= \sum_{i,k=1;\, i\neq k}^{n} \mathbf{E}_i^i \otimes \mathbf{E}_{-k}^{-k} + f(x^{-1}q)\sum_{i=1}^{n} \mathbf{E}_i^i \otimes \mathbf{E}_{-i}^{-i} \\
&\quad - g(xq^{-1}) \sum_{1\leq k<i\leq n} q^{k-i}\mathbf{E}_k^i \otimes \mathbf{E}_{-k}^{-i} + g(x^{-1}q) \sum_{1\leq i<k\leq n} q^{k-i}\mathbf{E}_k^i \otimes \mathbf{E}_{-k}^{-i} \\
\mathbf{R}_{1,2}^{(-,+)}(x) &= \sum_{i,k=1;\, i\neq k}^{n} \mathbf{E}_{-i}^{-i} \otimes \mathbf{E}_k^k + f(x^{-1}q^{-n-1})\sum_{i=1}^{n} \mathbf{E}_{-i}^{-i} \otimes \mathbf{E}_i^i \\
&\quad - g(xq^{n+1}) \sum_{1\leq i<k\leq n} q^{i-k}\mathbf{E}_{-k}^{-i} \otimes \mathbf{E}_k^i + g(x^{-1}q^{-n-1}) \sum_{1\leq k<i\leq n} q^{i-k}\mathbf{E}_{-k}^{-i} \otimes \mathbf{E}_k^i
\end{aligned}$$

is valid.

Proposition 1. If we define

$$\begin{aligned}
\tilde{\mathbf{R}}_{1,2}(x) &= \mathbf{R}_{1,2}^{(+,+)}(x) + \mathbf{R}_{1,2}^{(+,-)}(x) + \mathbf{R}_{1,2}^{(-,+)}(x) + \mathbf{R}_{1,2}^{(-,-)}(x) \\
\tilde{\mathbf{T}}(x) &= \mathbf{T}^{(+)}(x) + \mathbf{T}^{(-)}(x)\,,
\end{aligned}$$

the RTT–equation

$$\tilde{\mathbf{R}}_{1,2}(xy^{-1})\tilde{\mathbf{T}}_1(x)\tilde{\mathbf{T}}_2(y) = \tilde{\mathbf{T}}_2(y)\tilde{\mathbf{T}}_1(x)\tilde{\mathbf{R}}_{1,2}(xy^{-1})$$

is valid on the space $\mathcal{W}_0$.

Also, the R–matrix $\tilde{\mathbf{R}}(x)$ fulfills the Yang–Baxter equation

$$\tilde{\mathbf{R}}_{1,2}(x)\tilde{\mathbf{R}}_{1,3}(xy)\tilde{\mathbf{R}}_{2,3}(y) = \tilde{\mathbf{R}}_{2,3}(y)\tilde{\mathbf{R}}_{1,3}(xy)\tilde{\mathbf{R}}_{1,2}(x)$$

and has the inverse matrix

$$\big(\tilde{\mathbf{R}}_{1,2}(x)\big)^{-1} = \big(\mathbf{R}_{1,2}^{(+,+)}(x)\big)^{-1} + \big(\mathbf{R}_{1,2}^{(+,-)}(x)\big)^{-1} + \big(\mathbf{R}_{1,2}^{(-,+)}(x)\big)^{-1} + \big(\mathbf{R}_{1,2}^{(-,-)}(x)\big)^{-1}$$

where

$$\begin{aligned}
\big(\mathbf{R}^{(+,+)}_{1,2}(x)\big)^{-1} &= \frac{1}{f(x^{-1})}\Big(\sum_{i,k=1;\, i\neq k}^{n}\mathbf{E}^i_i\otimes\mathbf{E}^k_k + f(x^{-1})\sum_{i=1}^{n}\mathbf{E}^i_i\otimes\mathbf{E}^i_i \\
&\quad - g(x)\sum_{1\leq k<i\leq n}\mathbf{E}^i_k\otimes\mathbf{E}^k_i + g(x^{-1})\sum_{1\leq i<k\leq n}\mathbf{E}^i_k\otimes\mathbf{E}^k_i\Big) \\
\big(\mathbf{R}^{(-,-)}_{1,2}(x)\big)^{-1} &= \frac{1}{f(x^{-1})}\Big(\sum_{i,k=1;\, i\neq k}^{n}\mathbf{E}^{-i}_{-i}\otimes\mathbf{E}^{-k}_{-k} + f(x^{-1})\sum_{i=1}^{n}\mathbf{E}^{-i}_{-i}\otimes\mathbf{E}^{-i}_{-i} \\
&\quad - g(x)\sum_{1\leq i<k\leq n}\mathbf{E}^{-i}_{-k}\otimes\mathbf{E}^{-k}_{-i} + g(x^{-1})\sum_{1\leq k<i\leq n}\mathbf{E}^{-i}_{-k}\otimes\mathbf{E}^{-k}_{-i}\Big) \\
\big(\mathbf{R}^{(+,-)}_{1,2}(x)\big)^{-1} &= \sum_{i,k=1;\, i\neq k}^{n}\mathbf{E}^i_i\otimes\mathbf{E}^{-k}_{-k} + f(xq^{-n-1})\sum_{i=1}^{n}\mathbf{E}^i_i\otimes\mathbf{E}^{-i}_{-i} \\
&\quad + g(xq^{-n-1})\sum_{1\leq k<i\leq n}q^{i-k}\mathbf{E}^i_k\otimes\mathbf{E}^{-i}_{-k} - g(x^{-1}q^{n+1})\sum_{1\leq i<k\leq n}q^{i-k}\mathbf{E}^i_k\otimes\mathbf{E}^{-i}_{-k} \\
\big(\mathbf{R}^{(-,+)}_{1,2}(x)\big)^{-1} &= \sum_{i,k=1;\, i\neq k}^{n}\mathbf{E}^{-i}_{-i}\otimes\mathbf{E}^k_k + f(xq)\sum_{i=1}^{n}\mathbf{E}^{-i}_{-i}\otimes\mathbf{E}^i_i \\
&\quad + g(xq)\sum_{1\leq i<k\leq n}q^{k-i}\mathbf{E}^{-i}_{-k}\otimes\mathbf{E}^i_k - g(x^{-1}q^{-1})\sum_{1\leq k<i\leq n}q^{k-i}\mathbf{E}^{-i}_{-k}\otimes\mathbf{E}^i_k
\end{aligned}$$

The validity of the RTT–equation is Lemma 2. The Yang–Baxter equation that is equivalent to the equations

$$\mathbf{R}^{(\epsilon_1,\epsilon_2)}_{1,2}(x)\mathbf{R}^{(\epsilon_1,\epsilon_3)}_{1,3}(xy)\mathbf{R}^{(\epsilon_2,\epsilon_3)}_{2,3}(y) = \mathbf{R}^{(\epsilon_2,\epsilon_3)}_{2,3}(y)\mathbf{R}^{(\epsilon_1,\epsilon_3)}_{1,3}(xy)\mathbf{R}^{(\epsilon_1,\epsilon_2)}_{1,2}(x) \tag{2}$$

and the conditions for the inverse R–matrix, i.e. the relations

$$\mathbf{R}^{(\epsilon_1,\epsilon_2)}_{1,2}(x)\big(\mathbf{R}^{(\epsilon_1,\epsilon_2)}_{1,2}(x)\big)^{-1} = \mathbf{I}_{\epsilon_1}\otimes\mathbf{I}_{\epsilon_2}\,,\quad \text{where}\quad \mathbf{I}_{+} = \sum_{i=1}^{n}\mathbf{E}^i_i\,,\quad \mathbf{I}_{-} = \sum_{i=1}^{n}\mathbf{E}^{-i}_{-i}\,,$$

can be shown by direct calculation.

Definition. We denote the RTT–algebra defined by the R–matrix $\tilde{\mathbf{R}}(x)$ as $\tilde{\mathcal{A}}_n$.

We find out by the standard procedure from the RTT–equation (1) that in the RTT–algebra $\tilde{\mathcal{A}}_n$ mutually commutate not only the operators $\tilde{H}(x)$ and $\tilde{H}(y)$, where

$$\tilde{H}(x) = \mathrm{Tr}_{(+,-)}\big(\tilde{\mathbf{T}}(x)\big) = \mathrm{Tr}_{+}\big(\mathbf{T}^{(+)}(x)\big) + \mathrm{Tr}_{-}\big(\mathbf{T}^{(-)}(x)\big) = \sum_{i=1}^{n}\big(T^i_i(x) + T^{-i}_{-i}(x)\big)$$

but also all operators $\tilde{H}^{(\pm)}(x)$ a $\tilde{H}^{(\pm)}(y)$, where

$$\begin{aligned}
\tilde{H}^{(+)}(x) &= \mathrm{Tr}_{+}\big(\mathbf{T}^{(+)}(x)\big) = \sum_{i=1}^{n}T^i_i(x)\,,\\
\tilde{H}^{(-)}(x) &= \mathrm{Tr}_{-}\big(\mathbf{T}^{(-)}(x)\big) = \sum_{i=1}^{n}T^{-i}_{-i}(x)\,.
\end{aligned}$$

4 General Shape of Eigenvectors

Let $\mathbf{u} = (u_1, u_2, \dots, u_M)$ be an ordered set of mutually different complex numbers. We will look for eigenvectors in the form

$$\mathfrak{V}(\mathbf{u}) = \sum_{i_1,\dots,i_M,k_1,\dots,k_M=1}^{n} T^{i_1}_{-k_1}(u_1)T^{i_2}_{-k_2}(u_2)\dots T^{i_M}_{-k_M}(u_M)\mathit{\Phi}^{k_1,k_2,\dots,k_M}_{i_1,i_2,\cdots i_M}$$

where $\mathit{\Phi}^{k_1,k_2,\dots,k_M}_{i_1,i_2,\cdots i_M} \in \mathcal{W}_0$. Let us denote

$$\mathbf{B}(u) = \sum_{i,k=1}^{n} \mathbf{e}_i \otimes \mathbf{f}^{-k} \otimes T^{i}_{-k}(u) \in \mathcal{V}_+ \otimes \mathcal{V}_-^* \otimes \mathcal{A}$$

where $\mathbf{e}_i$ is the basis of the space $\mathcal{V}_+$ and $\mathbf{f}^{-k}$ is the basis of the space $\mathcal{V}_-^*$ and define

$$\begin{aligned}\mathbf{B}_{1,\dots,M}(\mathbf{u}) &= \mathbf{B}_1(u_1) \otimes \mathbf{B}_2(u_2) \otimes \dots \otimes \mathbf{B}_M(u_M)\\ &= \sum_{i_1,\dots,k_M}^{n} \mathbf{e}_{i_1} \otimes \dots \otimes \mathbf{e}_{i_M} \otimes \mathbf{f}^{-k_1} \otimes \dots \otimes \mathbf{f}^{-k_M} \otimes T^{i_1}_{-k_1}(u_1)\dots T^{i_M}_{-k_M}(u_M)\end{aligned}$$

If $\mathbf{f}^r$ is the dual basis with respect to $\mathbf{e}_i$ in the space $\mathcal{V}_+^*$ and $\mathbf{e}_{-s}$ is the dual basis with respect to $\mathbf{f}^{-k}$ in the space $\mathcal{V}_-$ and we denote

$$\mathbf{\Phi} = \sum_{r_1,\dots,r_M,s_1,\dots,s_M} \mathbf{f}^{r_1} \otimes \dots \otimes \mathbf{f}^{r_M} \otimes \mathbf{e}_{-s_1} \otimes \dots \otimes \mathbf{e}_{-s_M} \otimes \mathbf{\Phi}^{s_1,\dots,s_M}_{r_1,\dots,r_M}$$

we can write the general shape of Bethe vectors in the form

$$\mathfrak{V}(\mathbf{u}) = \Big\langle \mathbf{B}_{1,\dots,M}(\mathbf{u}), \mathbf{\Phi} \Big\rangle.$$

5 Commutation Relations $\mathbf{T}^{(\pm)}_0(x)\mathbf{B}_{1,\dots,M}(\mathbf{u})$

On the space $\mathcal{V}_0 \otimes \mathcal{V}^*_{1_+} \otimes \mathcal{V}_{1_-} \otimes \mathcal{A}$ we define

$$\widehat{\mathbf{T}}^{(+)}_{0;1}(x;u) = \left(\widehat{\mathbf{R}}^{(+,+)}_{0,1^*}(xu^{-1})\right)^{-1}\mathbf{T}^{(+)}_0(x)\widehat{\mathbf{R}}^{(+,-)}_{0,1}(xu^{-1})$$

$$\widehat{\mathbf{T}}^{(-)}_{0;1}(x;u) = \left(\widehat{\mathbf{R}}^{(-,+)}_{0,1^*}(xu^{-1})\right)^{-1}\mathbf{T}^{(-)}_0(x)\widehat{\mathbf{R}}^{(-,-)}_{0,1}(xu^{-1})$$

where

$$
\begin{aligned}
\big(\widehat{\mathbf{R}}^{(+,+)}_{0,1^*}(x)\big)^{-1} &= \frac{1}{f(x^{-1})}\Big(\sum_{i,k=1;\, i\neq k}^{n} \mathbf{E}^i_i \otimes \mathbf{F}^k_k \otimes \mathbf{I}_- + f(x^{-1})\sum_{i=1}^{n}\mathbf{E}^i_i \otimes \mathbf{F}^i_i \otimes \mathbf{I}_- \\
&\quad + g(x^{-1}) \sum_{1\le i<k\le n} \mathbf{E}^i_k \otimes \mathbf{F}^k_i \otimes \mathbf{I}_- - g(x) \sum_{1\le k<i\le n} \mathbf{E}^i_k \otimes \mathbf{F}^k_i \otimes \mathbf{I}_-\Big) \\
\big(\widehat{\mathbf{R}}^{(-,+)}_{0,1^*}(x)\big)^{-1} &= \sum_{i,k=1;\, i\neq k}^{n} \mathbf{E}^{-i}_{-i} \otimes \mathbf{F}^k_k \otimes \mathbf{I}_- + f(xq)\sum_{i=1}^{n}\mathbf{E}^{-i}_{-i} \otimes \mathbf{F}^i_i \otimes \mathbf{I}_- \\
&\quad + g(xq) \sum_{1\le i<k\le n} q^{k-i}\mathbf{E}^{-i}_{-k} \otimes \mathbf{F}^i_k \otimes \mathbf{I}_- \\
&\quad - g(x^{-1}q^{-1}) \sum_{1\le k<i\le n} q^{k-i}\mathbf{E}^{-i}_{-k} \otimes \mathbf{F}^i_k \otimes \mathbf{I}_- \\
\widehat{\mathbf{R}}^{(+,-)}_{0,1}(x) &= \sum_{i,k=1;\, i\neq k}^{n} \mathbf{E}^i_i \otimes \mathbf{I}^*_+ \otimes \mathbf{E}^{-k}_{-k} + f(x^{-1}q)\sum_{i=1}^{n}\mathbf{E}^i_i \otimes \mathbf{I}^*_+ \otimes \mathbf{E}^{-i}_{-i} \\
&\quad + g(x^{-1}q) \sum_{1\le i<k\le n} q^{k-i}\mathbf{E}^i_k \otimes \mathbf{I}^*_+ \otimes \mathbf{E}^{-i}_{-k} \\
&\quad - g(xq^{-1}) \sum_{1\le k<i\le n} q^{k-i}\mathbf{E}^i_k \otimes \mathbf{I}^*_+ \otimes \mathbf{E}^{-i}_{-k} \\
\widehat{\mathbf{R}}^{(-,-)}_{0,1}(x) &= \frac{1}{f(x)}\Big(\sum_{i,k=1;\, i\neq k}^{n} \mathbf{E}^{-i}_{-i} \otimes \mathbf{I}^*_+ \otimes \mathbf{E}^{-k}_{-k} + f(x)\sum_{i=1}^{n}\mathbf{E}^{-i}_{-i} \otimes \mathbf{I}^*_+ \otimes \mathbf{E}^{-i}_{-i} \\
&\quad + g(x) \sum_{1\le i<k\le n} \mathbf{E}^{-i}_{-k} \otimes \mathbf{I}^*_+ \otimes \mathbf{E}^{-k}_{-i} - g(x^{-1}) \sum_{1\le k<i\le n} \mathbf{E}^{-i}_{-k} \otimes \mathbf{I}^*_+ \otimes \mathbf{E}^{-k}_{-i}\Big)
\end{aligned}
$$

Lemma 3. In the RTT–algebra of $U_q\big(\mathrm{sp}(2n)\big)$ type the relations

$$
\begin{aligned}
\mathbf{T}^{(+)}_0(x)\Big\langle \mathbf{B}_1(u), \mathbf{f}^r \otimes \mathbf{e}_{-s}\Big\rangle &= f(x^{-1}u)\Big\langle \mathbf{B}_1(u), \widehat{\mathbf{T}}^{(+)}_{0;1}(x;u)\big(\mathbf{I}\otimes \mathbf{f}^r \otimes \mathbf{e}_{-s}\big)\Big\rangle \\
&\quad + g(xu^{-1})\Big\langle \mathbf{B}_1(x), \widehat{\mathbf{T}}^{(+)}_{0;1}(u;u)\big(\mathbf{I}\otimes \mathbf{f}^r \otimes \mathbf{e}_{-s}\big)\Big\rangle \\
\mathbf{T}^{(-)}_0(x)\Big\langle \mathbf{B}_1(u), \mathbf{f}^r \otimes \mathbf{e}_{-s}\Big\rangle &= f(xu^{-1})\Big\langle \mathbf{B}_1(u), \widehat{\mathbf{T}}^{(-)}_{0;1}(x;u)\big(\mathbf{I}\otimes \mathbf{f}^r \otimes \mathbf{e}_{-s}\big)\Big\rangle \\
&\quad - g(xu^{-1}))\Big\langle \mathbf{B}_1(x), \widehat{\mathbf{T}}^{(-)}_{0;1}(u;u)\big(\mathbf{I}\otimes \mathbf{f}^r \otimes \mathbf{e}_{-s}\big)\Big\rangle
\end{aligned}
$$

are valid.

For ordered M–tuples $\mathbf{u} = (u_1, \ldots, u_M)$, let $\overline{u}$ denote the set $\overline{u} = \{u_1, \ldots, u_M\}$. We define

$$
\begin{aligned}
\mathbf{u}_k &= (u_1, \ldots, u_{k-1}, u_{k+1}, \ldots, u_M)\,, \\
\overline{u}_k &= \overline{u} \setminus \{u_k\} = \{u_1, \ldots, u_{k-1}; u_{k+1}, \ldots, u_M\}\,, \\
F(x;\overline{u}^{-1}) &= \prod_{k=1}^{M} f(xu_k^{-1})\,, \qquad\qquad F(x^{-1},\overline{u}) = \prod_{k=1}^{M} f(x^{-1}u_k)\,.
\end{aligned}
$$

and introduce operators

$$\begin{aligned}
\widehat{\mathbf{T}}^{(+)}_{0;1,\dots,M}(x;\mathbf{u}) &= \Big(\widehat{\mathbf{R}}^{(+,+)}_{0,1^*}(xu_1^{-1})\Big)^{-1}\dots\Big(\widehat{\mathbf{R}}^{(+,+)}_{0,M^*}(xu_M^{-1})\Big)^{-1}\mathbf{T}^{(+)}_0(x)\\
&\qquad \widehat{\mathbf{R}}^{(+,-)}_{0,M}(xu_M^{-1})\dots\widehat{\mathbf{R}}^{(+,-)}_{0,1}(xu_1^{-1})\\
\widehat{\mathbf{T}}^{(-)}_{0;1,\dots,M}(x;\mathbf{u}) &= \Big(\widehat{\mathbf{R}}^{(-,+)}_{0,1^*}(xu_1^{-1})\Big)^{-1}\dots\Big(\widehat{\mathbf{R}}^{(-,+)}_{0,M^*}(xu_M^{-1})\Big)^{-1}\mathbf{T}^{(-)}_0(x)\\
&\qquad \widehat{\mathbf{R}}^{(-,-)}_{0,M}(xu_M^{-1})\dots\widehat{\mathbf{R}}^{(-,-)}_{0,1}(xu_1^{-1})\\
\mathbf{B}_{k;1,\dots,M}(x;\mathbf{u}_k) &= \mathbf{B}_k(x)\otimes\mathbf{B}_1(u_1)\otimes\dots\otimes\mathbf{B}_{k-1}(u_{k-1})\\
&\qquad \otimes\mathbf{B}_{k+1}(u_{k+1})\otimes\dots\otimes\mathbf{B}_M(u_M)
\end{aligned}$$

Proposition 2. The following relationships are applied:

$$\begin{aligned}
\mathbf{T}^{(+)}_0(x)\Big\langle\mathbf{B}_{1,\dots,M}(\mathbf{u}),\,\mathbf{\Phi}\Big\rangle &= F(x^{-1};\overline{u})\Big\langle\mathbf{B}_{1,\dots,M}(\mathbf{u}),\,\widehat{\mathbf{T}}^{(+)}_{0;1,\dots,M}(x;\mathbf{u})\mathbf{\Phi}\Big\rangle\\
&+\sum_{u_k\in\overline{u}} g(xu_k^{-1})F(u_k^{-1};\overline{u}_k)\Big\langle\mathbf{B}_{k;1,\dots,M}(x;\mathbf{u}_k),\\
&\qquad \Big(\widehat{\mathbf{R}}^{(+,+)}_{1^*,\dots,k^*}(\mathbf{u})\Big)^{-1}\widehat{\mathbf{R}}^{(-,-)}_{1,\dots,k}(\mathbf{u})\widehat{\mathbf{T}}^{(+)}_{0;1,\dots,M}(u_k;\mathbf{u})\mathbf{\Phi}\Big\rangle\\
\mathbf{T}^{(-)}_0(x)\Big\langle\mathbf{B}_{1,\dots,M}(\mathbf{u}),\,\mathbf{\Phi}\Big\rangle &= F(x;\overline{u}^{-1})\Big\langle\mathbf{B}_{1,\dots,M}(\mathbf{u}),\,\widehat{\mathbf{T}}^{(-)}_{0;1,\dots,M}(x;\mathbf{u})\mathbf{\Phi}\Big\rangle\\
&-\sum_{u_k\in\overline{u}} g(xu_k^{-1})F(u_k;\overline{u}_k^{-1})\Big\langle\mathbf{B}_{k;1,\dots,M}(x;\mathbf{u}_k),\\
&\qquad \Big(\widehat{\mathbf{R}}^{(+,+)}_{1^*,\dots,k^*}(\mathbf{u})\Big)^{-1}\widehat{\mathbf{R}}^{(-,-)}_{1,\dots,k}(\mathbf{u})\widehat{\mathbf{T}}^{(-)}_{0;1,\dots,M}(u_k;\mathbf{u})\mathbf{\Phi}\Big\rangle
\end{aligned}$$

where

$$\begin{aligned}
\widehat{\mathbf{R}}^{(+,+)}_{1^*,\dots,k^*}(\mathbf{u}) &= \widehat{\mathbf{R}}^{(+,+)}_{(k-1)^*,k^*}(u_{k-1}u_k^{-1})\dots\widehat{\mathbf{R}}^{(+,+)}_{2^*,k^*}(u_2u_k^{-1})\widehat{\mathbf{R}}^{(+,+)}_{1^*,k^*}(u_1u_k^{-1})\\
\widehat{\mathbf{R}}^{(-,-)}_{1,\dots,k}(\mathbf{u}) &= \widehat{\mathbf{R}}^{(-,-)}_{1,k}(u_1u_k^{-1})\widehat{\mathbf{R}}^{(-,-)}_{2,k}(u_2u_k^{-1})\dots\widehat{\mathbf{R}}^{(-,-)}_{k-1,k}(u_{k-1}u_k^{-1})\\
\widehat{\mathbf{R}}^{(+,+)}_{1^*,2^*}(x) &= \frac{1}{f(x)}\Big(\sum_{i,k=1;\,i\neq k}^{n}\mathbf{F}^i_i\otimes\mathbf{F}^k_k + f(x)\sum_{i=1}^{n}\mathbf{F}^i_i\otimes\mathbf{F}^i_i\\
&\qquad -g(x^{-1})\sum_{1\leq i<k\leq n}\mathbf{F}^i_k\otimes\mathbf{F}^k_i + g(x)\sum_{1\leq k<i\leq n}\mathbf{F}^i_k\otimes\mathbf{F}^k_i\Big)
\end{aligned}$$

6 Bethe Conditions and Eigenvectors of the Operator $H(x)$

Let us denote by $\widehat{T}^i_k(x;\mathbf{u})$ and $\widehat{T}^{-i}_{-k}(x;\mathbf{u})$ the operators defined by the relations

$$\begin{aligned}
\widehat{\mathbf{T}}^{(+)}_{0;1,\dots,M}(x;\mathbf{u}) &= \sum_{i,k=1}^{n}\mathbf{E}^k_i\otimes\widehat{T}^i_k(x;\mathbf{u})\,,\\
\widehat{\mathbf{T}}^{(-)}_{0;1,\dots,M}(x;\mathbf{u}) &= \sum_{i,k=1}^{n}\mathbf{E}^{-k}_{-i}\otimes\widehat{T}^{-i}_{-k}(x;\mathbf{u})\,.
\end{aligned}$$

The following statement, which gives part of the Bethe conditions, follows from the previous part.

Theorem 1. Let $\boldsymbol{\Phi}$ be common eigenvector of the operators

$$\begin{aligned}\widehat{H}^{(+)}_{1,\ldots,M}(x;\mathbf{u}) &= \mathrm{Tr}_0\Big(\widehat{\mathbf{T}}^{(+)}_{0;1,\ldots,M}(x;\mathbf{u})\Big),\\ \widehat{H}^{(-)}_{1,\ldots,M}(x;\mathbf{u}) &= \mathrm{Tr}_0\Big(\widehat{\mathbf{T}}^{(-)}_{0;1,\ldots,M}(x;\mathbf{u})\Big)\end{aligned}$$

with eigenvalues $\widehat{E}^{(+)}_{1,\ldots,M}(x;\mathbf{u})$ and $\widehat{E}^{(-)}_{1,\ldots,M}(x;\mathbf{u})$. If for each $u_k \in \overline{u}$ the relations

$$\widehat{E}^{(+)}_{1,\ldots,M}(u_k;\mathbf{u})F(u_k^{-1};\overline{u}_k) = \widehat{E}^{(-)}_{1,\ldots,M}(u_k;\mathbf{u})F(u_k;\overline{u}_k^{-1}) \tag{3}$$

are true, then $\Big\langle \mathbf{B}_{1,\ldots,M}(\mathbf{u}),\, \boldsymbol{\Phi}\Big\rangle$ is the eigenvector of the operator $H(x) = H^{(+)}(x) + H^{(-)}(x)$, where $H^{(\pm)}(x) = \mathrm{Tr}\big(\mathbf{T}^{(\pm)}_0(x)\big)$ with the eigenvalue

$$E_{1,\ldots,M}(x;\mathbf{u}) = \widehat{E}^{(+)}_{1,\ldots,M}(x;\mathbf{u})F(x^{-1};\overline{u}) + \widehat{E}^{(-)}_{1,\ldots,M}(x;\mathbf{u})F(x;\overline{u}^{-1})\,.$$

Thus, to find the eigenvectors of the operators $H(x)$, it is sufficient to find common eigenvectors of the operators $\widehat{H}^{(+)}_{1,\ldots,M}(x;\mathbf{u})$ and $\widehat{H}^{(-)}_{1,\ldots,M}(x;\mathbf{u})$.

Theorem 2. The operators $\widehat{\mathbf{T}}^{(\pm)}_{0;1,\ldots,M}(x;\mathbf{u})$ fulfill the RTT–equation

$$\begin{aligned}&\mathbf{R}^{(\epsilon,\epsilon')}_{0,0'}(xy^{-1})\widehat{\mathbf{T}}^{(\epsilon)}_{0;1,\ldots,M}(x;\mathbf{u})\widehat{\mathbf{T}}^{(\epsilon')}_{0';1,\ldots,M}(y;\mathbf{u})\\ &\qquad\qquad = \widehat{\mathbf{T}}^{(\epsilon')}_{0';1,\ldots,M}(y;\mathbf{u})\widehat{\mathbf{T}}^{(\epsilon)}_{0;1,\ldots,M}(x;\mathbf{u})\mathbf{R}^{(\epsilon,\epsilon')}_{0,0'}(xy^{-1})\end{aligned}$$

for any $\mathbf{u}$ and $\epsilon,\ \epsilon' = \pm$. Thus, they generate RTT–algebra $\tilde{A}_n$.

Theorem 3. The vector

$$\widehat{\Omega} = \underbrace{\mathbf{f}^1 \otimes \ldots \otimes \mathbf{f}^1}_{M\times} \otimes \underbrace{\mathbf{e}_{-1} \otimes \ldots \otimes \mathbf{e}_{-1}}_{M\times} \otimes \omega$$

is a vacuum vector for representation of the RTT–algebra $\tilde{\mathcal{A}}_n$ with the weights

$$\begin{aligned}\mu_1(x;\mathbf{u}) &= \lambda_1(x)F(x^{-1}q;\overline{u})\,,\\ \mu_{-1}(x;\mathbf{u}) &= \lambda_{-1}(x)F(xq;\overline{u}^{-1})\,,\\ \mu_k(x;\mathbf{u}) &= \lambda_k(x)F(xq^{-1};\overline{u}^{-1})\,, \quad k = 2,\ \ldots,\ n\,,\\ \mu_{-k}(x;\mathbf{u}) &= \lambda_{-k}(x)F(x^{-1}q^{-1};\overline{u})\,, k = 2,\ \ldots,\ n\,.\end{aligned}$$

So to find eigenvectors of the operators $H(x)$ for the RTT–algebra of $U_q\big(\mathrm{sp}(2n)\big)$ type, it is enough to formulate the Bethe ansatz for the RTT–algebra $\tilde{\mathcal{A}}_n$.

Acknowledgements The authors acknowledge financial support by the Ministry of Education, Youth and Sports of the Czech Republic, project no. CZ.02.1.01/0.0/0.0/16_019/0000778.

References

1. Faddeev, L.D., Sklyanin, E.K., Takhtajan, L.A.: Quantum inverse problem I. Theor. Math. Phys. **40**, 688–706 (1979)
2. Kulish, P.P., Reshetikhin, N.Yu.: Diagonalization of $GL(N)$ invariant transfer matrices and quantum N-wave system (Lee model). J. Phys. A. **16**(21983), L591–L596 (1983)
3. Burdík, Č., Navrátil, O.: Nested Bethe ansatz for RTT-algebra of so$(2n)$ type. Phys. Atom. Nucl. **81**(6), 810–814 (2018)
4. Burdík, Č., Navrátil, O.: Nested Bethe ansatz for RTT-algebra of sp$(2n)$ type. Phys. Part. Nucl. **69**(5), 936–942 (2018)
5. Burdík, Č., Navrátil, O.: Nested Bethe ansatz for RTT-algebra of sp(4) type. Theor. Math. Phys. **198**(1), 1–16 (2019)
6. Burdík, Č., Navrátil, O.: Nested Bethe ansatz for RTT–algebra of sp(4) type (2017). arXiv:1708.05633v1 [math-ph]
7. Reshetikhin, N.Yu.: Integrable models of Quantum one-dimensional magnets with $O(n)$ and $Sp(2k)$ symmetry. Teor. Mat. Fiz. **63**, 347 (1985)
8. Martins, M.J., Ramos, P.B.: The algebraic Bethe ansatz for rational braid-monoid lattice models. Nucl. Phys. B. **500**, 579–620 (1997)

Author Queries

Chapter 22

Query Refs.	Details Required	Author's response
AQ1	Per Springer style, both city and country names must be present in the affiliations. Accordingly, we have inserted the city name "Prague" in affiliation "2". Please check and confirm if the inserted city name is correct. If not, please provide us with the correct city name.	
AQ2	References [5] is given in the list but not cited in the text. Please cite them in text or delete them from the list.	

MARKED PROOF

Please correct and return this set

Please use the proof correction marks shown below for all alterations and corrections. If you wish to return your proof by fax you should ensure that all amendments are written clearly in dark ink and are made well within the page margins.

Instruction to printer	***Textual mark***	***Marginal mark***
Leave unchanged	• • • under matter to remain	(✓)
Insert in text the matter indicated in the margin	⋏	New matter followed by ⋏ or ⋏ⓧ2
Delete	/ through single character, rule or underline or ⊢—⊣ through all characters to be deleted	₰ or ₰ⓧ2
Substitute character or substitute part of one or more word(s)	/ through letter or ⊢—⊣ through characters	new character / or new characters /
Change to italics	— under matter to be changed	⎯⁄
Change to capitals	≡ under matter to be changed	≡
Change to small capitals	= under matter to be changed	=
Change to bold type	∼ under matter to be changed	∼
Change to bold italic	≂ under matter to be changed	≂
Change to lower case	Encircle matter to be changed	≢
Change italic to upright type	(As above)	⊹
Change bold to non-bold type	(As above)	⌇
Insert 'superior' character	/ through character or ⋏ where required	Y or X under character e.g. $\overset{2}{Y}$ or $\overset{2}{X}$
Insert 'inferior' character	(As above)	⋏ over character e.g. $\underset{2}{\wedge}$
Insert full stop	(As above)	⊙
Insert comma	(As above)	,
Insert single quotation marks	(As above)	$\overset{‘}{Y}$ or $\overset{‘}{X}$ and/or $\overset{’}{Y}$ or $\overset{’}{X}$
Insert double quotation marks	(As above)	$\overset{“}{Y}$ or $\overset{“}{X}$ and/or $\overset{”}{Y}$ or $\overset{”}{X}$
Insert hyphen	(As above)	⊢-⊣
Start new paragraph	_⌐	_⌐
No new paragraph	↩	↩
Transpose	⊔⊓	⊔⊓
Close up	linking ⁀‿ characters	⁀‿
Insert or substitute space between characters or words	/ through character or ⋏ where required	Y
Reduce space between characters or words	| between characters or words affected	ᛏ